Relativity In Logic

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RELATIVITY IN LOGIC

ONE of the most interesting suggestions made in connection with modern symbolic logic has been put forward by Prof. C I. Lewis. For him, there is not one but many possible systems of logic, each valid within its own domain. A somewhat similar attitude towards logic, though based on different grounds, has been expressed by Santayana in his recent *Realms of Essence.*

It is by the discussion of the primitive ideas of (1) *implication* and (2) *truth and falsity* that I shall try to show some grounds for this contention. In the first part of the paper a distinction between two fundamental types of logics will be made. In the second part the necessity for a propositional analysis of logic will be discussed, and a justification for the generalization of the concepts *proposition* and *truth* and *falsity* will be advanced. In the third part a schema will be presented for the ready apprehension of the nature of one of the logics discussed in part one, and a wide variation in the interpretation of the kind of implication that it employs will be there indicated.

I

In his discussion of the logic of internal and external relations,[1] G. E. Moore made a distinction between an internal and necessary connection, and an external or factual one, using the term "entails" to express the former, and

[1] *Cf.* "External and Internal Relations" in *Arist. Soc. Proc.,* 1919-20, p. 53.

"implies" to express the latter. The distinction had already been recognized by Whitehead and Russell in the first edition of their work (Principia Mathematica) though a distinction in terminology was not made. It is there said[2] that the assertion sign before the propositions is required "for distinguishing a complete proposition which we assert, from any subordinate propositions contained in it but not asserted." It was only the asserted propositions which were said to be true and certifiable on logical grounds alone. Using Moore's terms we shall say that the asserted propositions involve an entailment, while those propositions which are only "considered" involve an implication.

Such a distinction between implication and entailment is not on all fours with a distinction between contingent and necessary relations. Though the considered propositions involve an empirical connection between the constituents, each of which must be known separately and extralogically, while the asserted propositions depend on no empirical data for their validity, both may be necessary. When a proposition is asserted, an entailment is used to connect truth-value or truth-functions[3] which are in some sense interdependent; when considered, an implication is used to connect entities which *may be* interdependent. Though one might not be able to say in any given situation whether it contained a necessary relation, such empirical data are indifferent to its presence. Such a relation might be represented by what Prof Lewis calls a strict implication[4] and when employed in a logic would itself have to

[2] p. 8.

[3] A truth function is a function of propositions whose truth or falsity depends only upon the truth or falsity of the constituent propositions. (*Prin. Math.*, p. 8.) Thus 'P and Q' is a truth function of P and of Q, and is true if both are true, and false otherwise. 'P or Q' is true if either P is true or Q is true and is false otherwise. The truth-value of a proposition is defined (p. 7) as truth if the proposition is true and falsehood otherwise.

[4] *Survey of Symbolic Logic*, p. 293.

The truth-value of a proposition is defined in the *Principia* (p. 7) as truth if the proposition is true and falsehood otherwise.

be distinguished from strict entailment. Though both expressed necessary connections it would be only the latter which would be treated in a strictly formal logic, and asserted without recourse to experience or extra-logical knowledge. There are thus four ideas to be distinguished: (1) implication or a correlation between independent entities; (2) entailment, or a correlation between truth values or truth functions to yield a tautology; (3) strict implication, or a necessary factual correlation between entities; and (4) strict entailment, or a necessary correlation between truth functions,[5] to yield tautologies.

As has been pointed out by Prof. Lewis,[6] the use of the term implication in sense (1) is somewhat strained. In ordinary parlance implication means 'necessitates,' denoting that "relation which is present when we 'validly' pass from one assertion, or set of assertions to another assertion, without any reference to additional 'evidence'." If it be true that there is a 'proper' meaning of the word, and that this is its proper meaning, then a symbolic logic which uses a different one, though it may be internally coherent, will not yield a criterion of ordinary inference.[7] If it achieved its theorems by the employment of postulated methods of substitution in the postulates or their derivatives, or by the consistent use of its arbitrary notion of inference on these postulates, it would be a system which would be unexceptional on its own grounds, but not necessarily applicable to what is usually considered as trains of reasoning.

If it be recognized that a logical system need not have

[5] There is a further distinction between implication as functioning between individual elements and between classes or groups of them—the latter being a "formal" implication. The formal implication, however, is a derivative from (1) or (3) and throws no light on our discussion.

[6] *Survey,* p. 324.

[7] See also W. E. Johnson's discussion of the meaning of the paradox that a false proposition implies any. *Logic,* Vol. 1, p. 45.

an application and that non-applicable systems are as much the logician's study as the different possible geometries are the mathematician's, it will soon be evident that there is more than one such system. However, just as non-Euclidean geometries are not distinguished from the Euclidean on the basis of their applicability or lack of applicability to our perceptual or physical space, the different systems of logic should be differentiated on some other ground than their possible application to our usual 'legitimate' trains of reasoning. It should be indifferent to a logician whether his system can be used. The differentiation between the kinds of implications makes it possible to speak of "accidental" and "necessary" logics. It is to the former class that the system developed in the *Principia* belongs. It, like the other systems to be mentioned in part three, makes the assumption that a given proposition has properties which do not affect the properties of any other proposition. By following this idea through consistently Wittgenstein developed a metaphysics where "superstition was the belief in the causal nexus" and where any one fact "can either be the case or not be the case and everything else remain the same." All relations for Wittgenstein, apparently, are external and contingent.

I have so far distinguished a correlation between facts, from a correlation between truth-values and/or truth-functions. I have pointed out the possibility of differences in the interpretation of implication, using the notions of necessity and contingency as fundamental, thus securing two types of logic—"necessary" and "accidental." Lewis' and MacColl's systems are in the former class; the *Principia* is in the latter. No disparagement is involved in the distribution of the names.

It has often been contended that strict entailment is simply formal implication or that it is identical with entailment as employed in the *Principia.* If this be true, it would not involve the elimination of strict *implication* for there is a difference between a factual relation between independent elements and a factual relation between interdependent ones, and thus between material and strict implication. It would make the only difference between these logics consist in their applicational range—one applying to necessary facts, the other to contingent.

II

The systems of both Whitehead and Russell and of Lewis treat of (1) the truth and falsity of (2) propositions and what is derived from propositions by a process of generalization. In Lewis' system (as well as that of MacColl's),[8] a third truth-value is introduced, that of impossibility, which when combined with the other two gives an infinite number of possible truth-values. The accidental propositional logics, so far developed, all restrict themselves to only two truth-values, defined as mutually exclusive,[9] expressing the truth or falsity of any proposition treated. They are what C. S. Peirce would call dichotomic logics.

There is no necessity for a systematic logic to restrict itself to two truth-values, or if only two are considered, to the two values, truth and falsity. Passing over the first point, which is one of the bases for the systems of MacColl and Lewis, and referring only to the two truth-values accepted in the other systems, it should be evident that any two properties defined as mutually exclusive would do just as well as truth and falsity, for the purposes of a symbolic logic. The truth that concerns logic is not the

[8] *Symbolic Logic and its Applications*, p. 6.

[9] It is one of the paradoxes of modern logic that one of the exclusive classes may be contained in the other.

truth of its elements but the truth of a tautology. In a system where we dealt only with say, reality and non-existence, and where any of the four possible combinations of a real with a non-existent entity would itself be real or non-existent, a perfectly definite dichotomic logic in which these were tautologically correlated would be possible. The entailment of any of the entities by any of the others would of course make it possible to say, "it is *true* that such and such a connection holds." That statement would not be part of the connection any more than the assertion sign is part of what is asserted. Thus Russell in "Appendix C" of the second edition of the *Principia:* "When we say that truth or falsehood is for logic the essential characteristic of propositions we must not be misunderstood. It does not matter for mathematical logic, what constitutes truth or falsehood; all that matters is that they divide propositions into two classes according to certain rules."[10] "We are concerned only with those combinations of propositions which are true in virtue of the rules, whether their constituent propositions are true or false."[11] Inasmuch as it doesn't make any difference what constitutes truth and falsity, or what truth-value a constituent has, as long as there are two classes and tautologies are possible, one should be able to employ variable notions instead of constant ones for truth-value and truth-function. Accordingly, I shall substitute a variable for truth-value. and get *property-value,* which is a variable for the category in which every entity in the logic either is or is not. General logics may be distinguished in terms of the values they assign to this variable. In the same way, for truth-function, we substitute a variable and secure *compound.* A *categorical value* would be the value of any complex of

[10] p. 660.
[11] p. 661.

entities,[12] combined through the medium of logical constants alone, which value was a function of the property-values of its constituents. It is not necessary that a categorical-value be of the same kind as the property-values (or their contradictories) of the constituents of the compound. Thus the combination of an impossible with a necessary entity might yield a compound whose categorical value was falsity.

Truth and falsity have been considered as fundamental logical property-values primarily because the basic unit of language has been taken to be a proposition. Only propositions are said to be true or false, and as the propositional analysis seems to make possible the clarification of mathematical concepts, which is the express purpose of a symbolic system, a true-false propositional logic was inevitable. But if a dichotomic or two-valued logic requires nothing more than two categories to distinguish its elements, the insistence on the propositions as the fundamental units should disappear with the denial of an exclusive interest in truth and falsity.[13] There must, of course, be propositions *of* logic, for logic in one sense is a language, but these propositions *of* logic need not refer to propositions or only to certain characters of them. Logic might very well start with an analysis of the relations of essences to one another, and the different logics could be different tautological dialectics of the realm of essence. A logic after all does not have to say, "if this is *true,* then tautologically that is *true,* but simply "if such and such, then tautologically such and such." Statements of both kinds are propositions *of* logic and have logical truth. The truth of correspondence, which may be said to be the character of a

[12] I anticipate the substitution of "entity" for proposition to express the content of the system.

[13] This is not an insistence on a return to the Boole-Schroeder algebra but a plea for an analysis in terms of other entities than language or "thought" units.

non-logical proposition is only one of many properties that can be correlated in such logical propositions. We do have a habit of speaking of the "truth that this is such and such" but the distinction between the such and such, and the truth should be apparent. It is the *such and such* that should be emphasized and not the *truth* that it is a such and such though truth itself could be one of the such and suches. The fact of Washington's crossing the Delaware which has nothing to do with correspondence, propositions, judgment or assertion, excludes other f a c t s . Though we may not be able to deal with it except in terms of a proposition, such an application should be irrelevant to a science interested in mapping out abstract possibilities. By holding to the proposition as the unit for logic, Wittgenstein was compelled to view it as a picture of facts, whose relation to them was unstatable.

By substituting variables for truth-value, truth-function and proposition, we secure property-value, compound categorical-value and elements. In accordance with the values which any systems impose on these variable different specific applications of logic are secured. A general logic would be one of a group of logics with the same number of property-values and categorical values all of which were specific determinations of these variables. The *Principia,* insofar as one is interested in a two-valued logic, might still serve as a general form for the entire class of such logics, simply by ignoring the entire introduction. The Introduction would be an obiter dicta, expressing two logicians' fancy.

III

Taking any two elements and two property-values, their combination yields four basic compounds. We restrict ourselves to two elements for convenience's sake, though what is said of two will hold of any number. Two property-values only are dealt with, for that will be sufficient to illustrate the fact that a number of logics are possible. To enable a ready comparison with the *Principia* P and Q will be used to symbolize elements, and T and F the property-values, though as has been indicated, the elements need not be propositions, nor the values truth and falsity. The four compounds are: (1) both T and T; (2) and (3) one T and the other F; and (4) both F, expressed in the following table:

P	*Q*	*Compounds*
T	T	P. Q
T	F	P.—Q
F	T	—P. Q
F	F	—P.—Q

Thus when P is T, and Q is F, we have the compound P.—Q, with—Q as the negative of Q. (The dot represents conjunction.) By making selections from this set of four compounds sixteen combinations of compounds are possible. Every logical combination of two elements, as well as each element individually, is expressed by one of the sixteen selections from this set.

A much simpler way of building up these various compounds has been indicated by Prof. Sheffer.[14] He takes as a primitive idea 'p is incompatible with q' expressed as p/q. By substituting p for q, and q for p, we get p/p and q/q. Substituting either one of these or both for p or q in the primitive idea, we secure four compounds—illustrating what is meant in the *Principia* by negation, implication, disjunction and c o n j u n c t i o n — p/p; p/(q/q); (p/p)/(q/q) and (p/q)/(p/q). By proper substitutions in these four all the compounds with p and q can be developed, so that only one primitive idea and the principle

[14] *Cf. Trans. Amer. Math. Soc.,* Vol. XIV, p. 481.

of substitution are necessary in order to secure all the compounds in that system. This method, though symbolically convenient, however, does not show clearly the interrelationship between the different compounds nor indicate the difference between implication and entailment; and as the purpose of this paper is exposition rather than the achievement of symbolic efficiency, I shall use a more cumbersome but psychologically more evident method.[15]

We first assert all the four compounds disjunctively and deny in turn first, each one, then combinations of two, combinations of three and finally all, thus securing the following set. This is a schema which has been employed by Wittgenstein[16] though some changes in the arrangement have been made to bring out the symmetry.

		TRUTH FUNCTION				Denies
(A)	TTTT	P.Q v	P.-Q v	-P.Q v	-P.-Q	(Z)
(B)	FTTT		p.-q	-p.q	-p.-q	(O)
(C)	TFTT	p.q		-p.q	-p.-q	(N)
(D)	TTFT	p.q	p.-q		-p.-q	(M)
(E)	TTTF	p.q	p.-q	-p.q		(L)
(F)	FTTF		p.-q	-p.q		(K)
(G)	FFTT			-p.q	-p.-q	(J)
(H)	FTFT		p.-q		-p.-q	(I)
(I)	TFTF	p.q		-p.q		(H)
(J)	TTFF	p.q	p.-q			(G)
(K)	TFFT	p.q			-p.-q	(F)
(L)	FFFT				-p.-q	(E)
(M)	FFTF			-p.q		(D)
(N)	FTFF		p.-q			(C)
(O)	TFFF	p.q				(B)
(Z)	FFFF	-(p.q).-(p.-q).-(-p.q).-(-p.-q)				(A)

15 Russell in the introduction to the second edition employs Sheffer's stroke-function, but as the derivation of the various truth-functions from p/q was not made entirely clear, I have thought it best to state it explicitly.

16 *Cf. Tractatus,* 5:101.

The definition of implication as employed in the *Principia Mathematica* is expressed by (C); material implication being simply the statement that one or the other of the three compounds holds and that (N) or p.q fails. Whenever any set of compounds is contained in the others, the first is said to entail the others. Thus (J) which says that p holds is contained in (A), (D), and (E), so that p entils 'por -p', 'q or -q', 'q implies p,' and 'p or q.' As (J) contains (N), (O), and (Z), 'p-q,' 'p.q,' 'p.-p,' and 'q.-q' entail p, and as it contains itself, p entails p. The propositions of this logic are thus seen to be tautologies, for the entailment unites a part of a disjunctive set with the whole of the set. Or more evidently, if the part be denied and its contradictory disjoined with the whole we get (A). Thus (J) entails (D); -J or G equals (-P.Q v -P. -Q) (i. e. not P) which when disjoined with (A), (D), or (E) is (A) or a redundant expression of it.

If instead of taking (C) as our definition of implication we had taken (B), in exchange for the paradoxes of the *Principia* where a false proposition implies any, and a true proposition is implied by any, (interpreted propositionally), we would get: a false proposition implies and is implied by any; a true proposition implies and is implied only by a false proposition. The entailment is the relation of a truth-function with one which does not contain it, so that (J) entails (G), (L), (M) and (Z), i. e., p entails '-p,' '-p.-q,' '-p.q,' '-p.q' and '-q.q.' The rule of inference in the system is: if p is true, and p implies q, then q is false. It is to be noted that the theorem: —(P entails —P) entails that —(P entails —P) entails (P entails —P), holds, which is to say that if any theorem in the system be denied then that denial entails that the denial entails the truth of the theorem. The logic is thus internally coherent.

By varying the definition of implication fourteen systems are possible. (A) or tautology is eliminated because it permits all connection and (Z) or contradiction because it permits none. Each of these systems is self-consistent and tautological and thus a "logic" being capable of statement in terms of variable elements and logical constants alone. Most of these systems will have no use, as they involve connections which are unfamiliar. In addition, all of these systems can be related to one another by means of a transformation formula, for what is implication in one of the systems is simply equivalence, disjunction, negation, etc., in the other. (B) for example in system (C) is (-P v -Q). Which one should be taken as basic is a matter of extra-logical choice.

Relativity in logic is thus seen to be relativity in notation or in the definition of implication. The fact that the familiar laws of logic (contradiction, excluded middle, etc.) are invoked to permit the discrimination between elements, values, functions, etc., points to an absolute logic over and above the symbolic statements. One can never include in the system the principles of conditions which determine it. This is one of the real meanings of the theory of types,[17] and is as old as Aristotle. Relativity in logic does not mean, therefore, a Schillerian scepticism, but simply a freedom in choice of expression.

[17] *Cf. "The Theory of Types," Mind,* July, 1928, p. 344.

Conclusion

Implication, when not defined as a necessary relation holding in any context, may be defined as any one of fourteen correlations between the members of two classes, or more accurately, between elements which have one or the other of two property-values. This takes care of the 'accidental' logics only. Entailment functions between part and whole. Truth and falsity are only two of an indefinite range of properties which can be employed, while the proposition is only one of an indefinite range of elements. A variation in the values for the implication, categorical-value, and property-value, gives a wide range of possible applied or general logics. A logic which includes more than two properties may contain a two-valued logic. The *Logic* has an infinite number of elements capable of an infinite number of property-values. It can never be expressed.

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